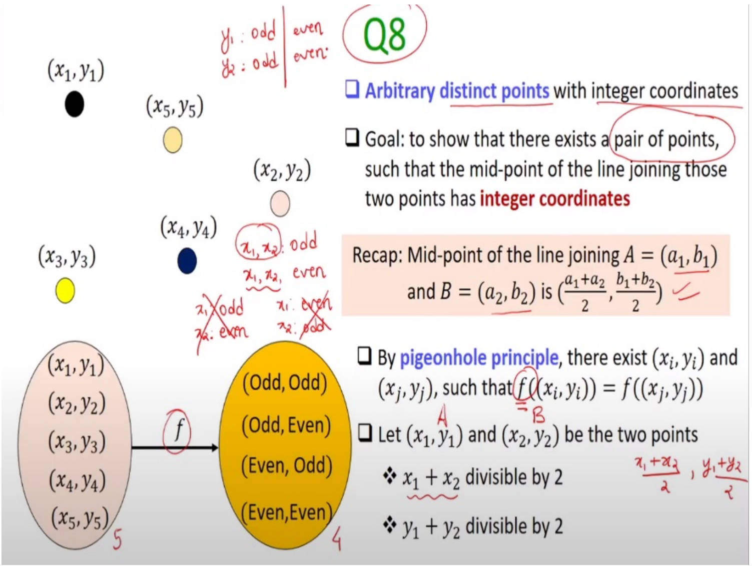
Discrete Mathematics
Prof. Ashish Choudhry
IIIT, Bangalore

Module No # 08
Lecture No # 38
Tutorial 6: Part II

Hello everyone, welcome to the second part of tutorial 6.

(Refer Slide Time: 00:24)



Let us start with question number 8. You are given here arbitrary distinct points in 2 dimensional planes. Each point will have an x-coordinate, y-coordinate and the points are having integer coordinates. So they are arbitrary points except that they are distinct. So, I am denoting the points, their respective coordinates as $x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4$ and x_5, y_5. And our goal is to show that irrespective of the way these 5 points are chosen arbitrarily they are always exist a pair of points such that if you consider the midpoint of the line joining those 2 points it has integer coordinates.

So just to recap if you have 2 points, a point with coordinates (a_1, b_1) and another point with coordinates (a_2, b_2) then the midpoint of the line joining these 2 points is given by the formula $\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right)$. And we want to apply here pigeonhole principle. So remember for pigeonhole principle we have to identify the set of pigeons and the set of holes here and then the mapping which relates the pigeon and the holes.

So let us do that. So consider the set of 5 arbitrary points which are all distinct and have integer coordinates. We are trying to map this point depending upon what is the nature of their x-coordinate and y-coordinate. So depending upon whether the x-coordinate is even, or x-coordinate are odd, or whether the y-coordinate is odd, or the y coordinate is even, I have 4 possible combinations.

And my function f maps these 5 points to the corresponding pair; say if x_1 is odd and y_1 is even then I will say that $f(x_1, y_1)$ is (odd, even) and so on. That is the mapping here. So now, it follows from pigeonhole principle that we have now 5 items here in the set A and 4 items in the set B then there always exist a pair of points among these 5 points say (x_i, y_i) and (x_j, y_j) such that both of them are mapped to the same ordered pair.

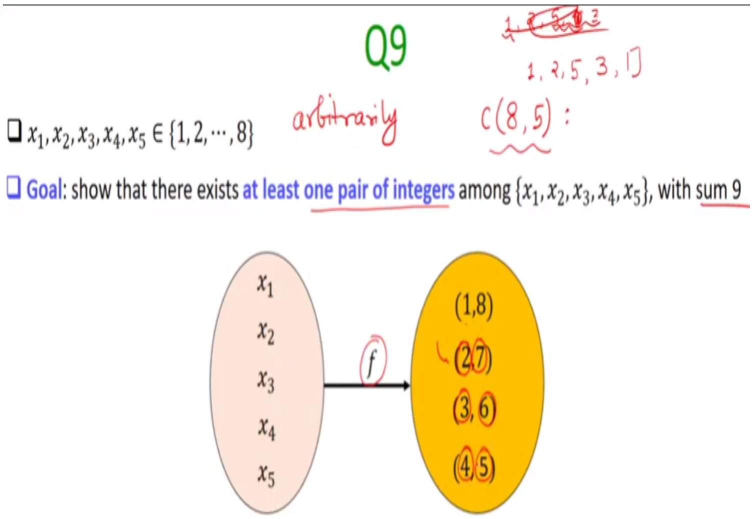
So it could be any 2 out of those 5 points; it could be the first 2 points, it could be the last 2 points, it could be the third point or the fourth point and so on; we do not know. It depends upon the exact 5 points that we chose. So, without loss of generality assume that out of those 2 points which are guaranteed to be mapped to the same ordered pair are the first 2 points.

So say (x_1, y_1) and (x_2, y_2) be the 2 points such that the corresponding f output of the f function for these points are the same. Now we want to inspect what happens to the midpoint of the line joining these 2 points A and B. So as per the formula the midpoint of the lines joining these 2 points A and B will be $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

And since both the points A and B are mapped to the same ordered pair; so for instance it could be the case that both x_1 as well as x_2 are odd or it could be the case that both x_1 as well as x_2 are even. So irrespective of the case $x_1 + x_2$ will always be divisible by 2. If both of them are even definitely sum of 2 even quantities is divisible by 2. Whereas if both of them are odd then also the sum of 2 odd quantities is divisible by 2. And as per our assumption it is not the case that x_1 is odd and x_2 is even that is not the case because we are considering the case when the output of the f function on these 2 points A and B are the same.

In the same way we cannot have the case where x_1 is even and x_2 is odd because that is not the property of the point A and B. Due to the exactly the same reason, the type of y_1 and y_2 coordinates are the same. Either they are both odd or both of them are even right. And again in this case it is easy to see that $y_1 + y_2$ will be divisible by 2. And that shows that this statement is a correct statement.

(Refer Slide Time: 05:22)



So let us go to question number 9. Here you are given the following. You are choosing 5 integers from the set 1 to 8 arbitrarily. Our goal is to show irrespective of the way you choose those 5 points there always exists at least one pair of integers among those chosen 5 integers whose sum is 9. So say you pick 1, 2 and 5 and then if you pick 3 then you still do not have any pair of integers whose sum is 9. But as soon as you pick the fourth point, so if you pick 4 that is the fifth number then you have 5 and 4 which is summing up to 9.

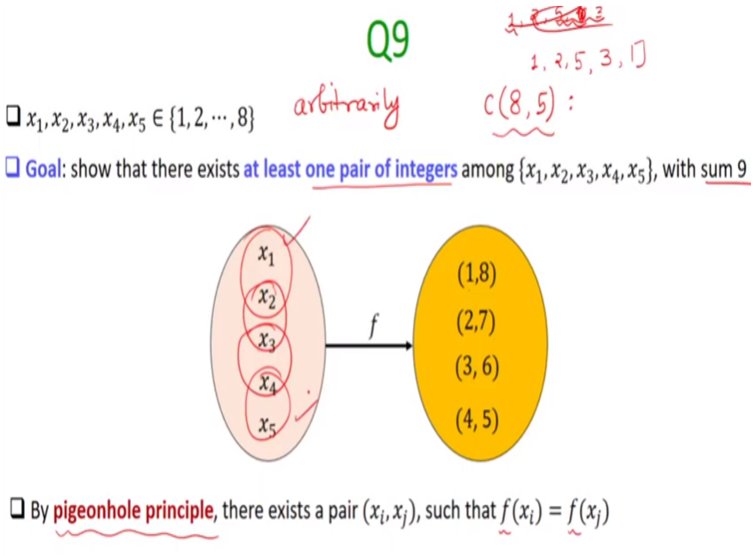
If you pick 6 as the fifth number, then you have 3 and 6 summing up to 9. If you have if you pick 7 as the fifth number, then you have 7 and 2 summing up to 9. If you pick 8 as the fifth number, then you have 1 and 8 summing up to 9 and so on. So you can verify this by an example but we want to prove it irrespective of the 5 numbers that we are going to pick.

So one way of proving this is that you take all possible 8 choose 5 ways of picking 5 numbers and for each of those combinations you show that the statement is true but that will be an overkill because this is a relatively large value. Instead we will apply the pigeonhole principle and again for applying the pigeonhole principle we have to identify the pigeons and the holes and the mapping. So my pigeons here are the 5 integers among the numbers 1 to 8 that I am picking arbitrarily and my holes are the ordered pairs of distinct integers in the set 1 to 8 whose sum will give you 9.

So you have either the ordered pair (1, 8) or the ordered pair (2, 7) or ordered pair (3, 6) or the ordered pair (4, 5). And you do not have any other ordered pair from the set 1 to 8 summing up to 9. And my function f basically maps these x_i values to the corresponding ordered pair depending upon whether x_1 is 1 or 8 I will say that $f(x_1)$ is either $(1,8)$. Or if x_1 takes either the value of 2 or the value of 7 then I will say $f(x_1)$ is (2, 7).

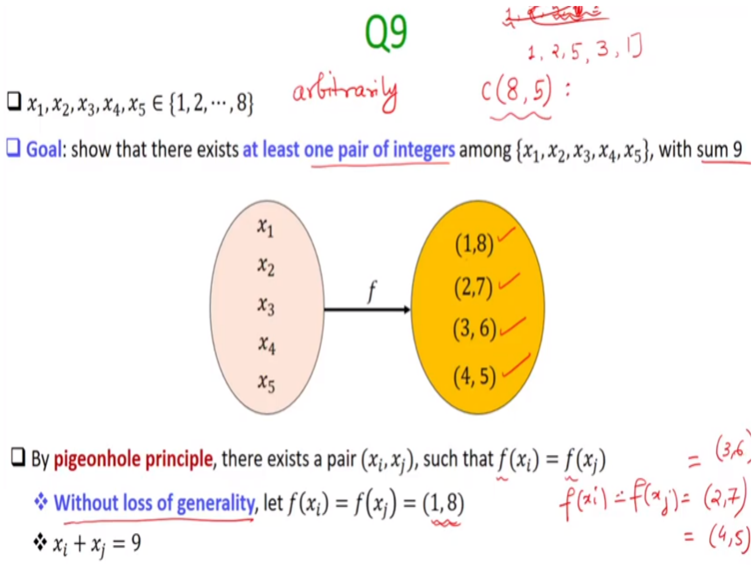
Or if my x_1 is either 3 or 6 then I will say that $f(x_1)$ is (3, 6) or if my x_1 is either 4 or x_1 is either 5 then I will say that $f(x_1)$ is (4, 5). That is the interpretation for my mapping f.

(Refer Slide Time: 08:51)



Now it follows simply from pigeon-hole principle that there always exists a pair or two values out of the 5 numbers say (x_i, x_j) such that $f(x_i)$ and $f(x_j)$ are the same. It could be say the first 2 values, the last 2 values, the second or the third value, the third or the fourth value, or the first value or the fifth value; it could be any 2 values out of those 5 numbers.

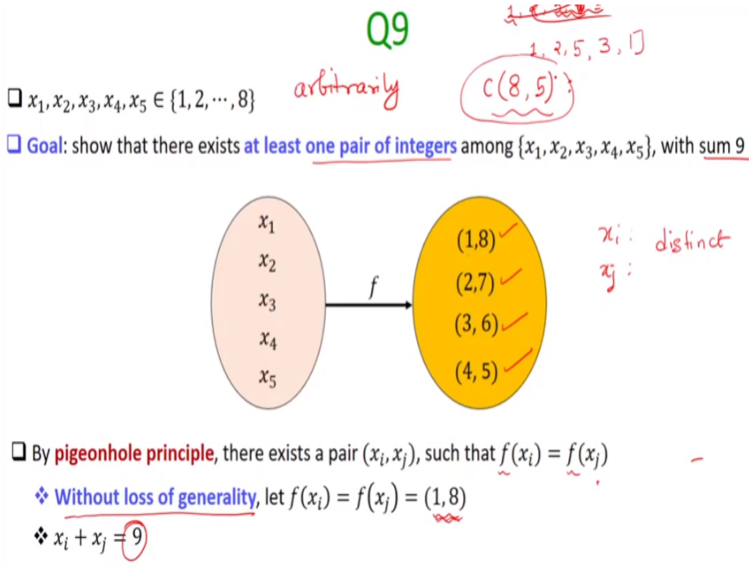
(Refer Slide Time: 09:24)



We do not know which one. So again without loss of generality, suppose both of them got mapped to (1, 8); we do not know what is the identity of x_i or x_j and we do not know the corresponding mapping as well. It could be either (1, 8), (2, 7), (3, 6) or (4, 5). So, without loss of generality; that means whatever reasoning we are giving here for the case where $f(x_i) = f(x_j) = (1,8)$ hold, the same argument will hold even if $f(x_i)$ is same as $f(x_j)$ is equal to say (2, 7); the same reasoning will hold symmetrically for that case as well.

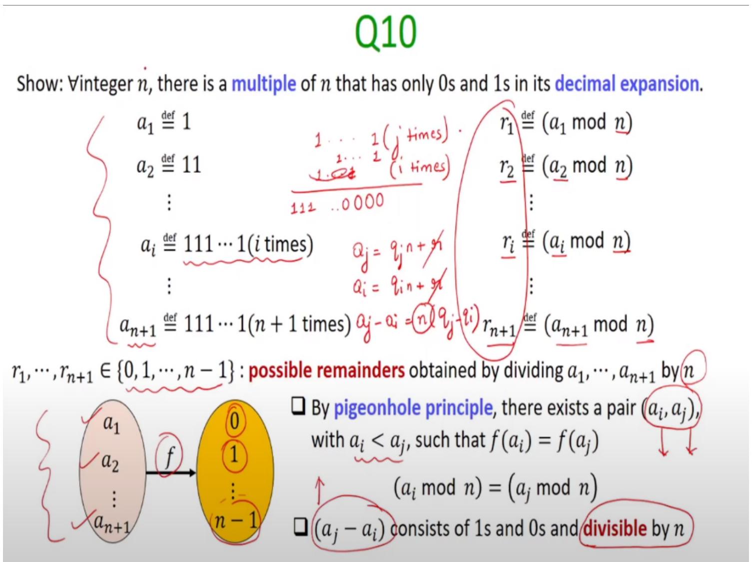
Symmetrically for the case when it is (4, 5), symmetrically for the case when it is (3, 6) and so on. So that is why we do not consider the remaining 3 cases. We just consider the case when $f(x_i)$ and $f(x_j)$ is (1, 8).

(Refer Slide Time: 10:21)



If that is the case then since your x_i and x_j are distinct and they got mapped to (1, 8) that means either x_i is 1 and x_j is 8 or x_i is 8 and x_j is 1. Irrespective of the case, the sum of x_i and x_j is 9. So now you can see that even without enumerating all possible $C(8,5)$ arrangements or combinations of picking 5 numbers out of these 8 numbers we ended up arguing in a very simple fashion that our statement is true using pigeonhole principle. It shows the power of this proof strategy or counting mechanism basically.

(Refer Slide Time: 11:07)



So question 10 we want to prove a universally quantified statement. Namely, we want to prove that you take any integer n, there is always a multiple of n which has only the digits 0's and 1's in its decimal expansion. So before going into the proof if you want to take few examples say $n = 1$ then I always have the number 1 which is a multiple of 1 and which has only the digit 1 in its decimal expansion.

Remember it is not mandatory that you have both 0's as well as 1 in the decimal expansion. The only restriction is we have to show that in the decimal expansion you only have either the digits 0's or 1's. If you take $n = 2$ then I can take the number 10 which is a multiple of 2 and which has only 1's and 0's and in its decimal expansion. If I take $n = 3$ then I can take the number 111 which has only the digit 1 in its decimal expansion and which is divisible by 3.

So at least by taking few examples we found that the statement is true. But this is a universally quantified statement and we cannot prove a universally quantified statement just showing examples for a few cases. So we have to give the proof for arbitrary n. Again, we are going to apply here pigeonhole principle. So let me define a few decimal numbers here.

I define the first decimal number to be 1. I define second decimal number as 11, the i-th decimal number as a decimal number consisting of i number of 1's and the $n + 1$ decimal number which has the digit 1, $n + 1$ number of times. Let me define another set of values. So my value r_1 is the remainder which I obtain by dividing a_1 by n. Similarly, I define r_2 to be the remainder obtained by dividing a_2 by n. I define r_i to be the remainder obtained by dividing a_i by n.

And in the same way I define r_{n+1} as the remainder obtained by dividing a_{n+1} by n. Now what can I say about this remainders? It is easy to see that these remainders belong to the set 0 to $n - 1$ because of the simple fact that you divide any number by n the only possible remainders could be 0 if it is completely divisible by n or the remainders could be $1, 2$ ... $n - 1$. Now you have to apply the pigeonhole principle.

So my pigeons are the numbers a_1 to a_{n+1} that I have constructed here. And my holes are basically the remainders which I can obtain by dividing these $n + 1$ numbers by n. And I have n possible remainders and my function f map the numbers to the corresponding remainder which I have obtained by dividing that number by n. So you have more number of numbers and less number of remainders.

So it follows from the pigeonhole principle that you always have a pair of numbers a_i and a_j out of this $n + 1$ numbers which gives you the same remainder if you divide a_i and a_j by n. I do not know the remainder it could be either 0, or the remainder could be either 1, or the remainder could be $n - 1$.

I do not know what are the individual remainders that a_i and a_j are going to give on dividing by n. But what I know is that they are leaving the same remainder. And again without loss of generality assume that a_i is occurring before a_j in my sequence here. Now what can I say about this number $a_j - a_i$. So a_j will be a number which has j number of 1's and a_i is another number which has i number of 1's. Both of them gives me the same reminder on dividing by n.

So if I take $a_j - a_i$then this will be a decimal number which will have trailing 0's and then at the leading positions you will have the 1's. That means it is a decimal number which has only the characters 1s and 0's. But what can you say about its divisibility by n. This number will be divisible completely by n because a_j gives you the same remainder, say r, so I can say a_j is some $q_j * n + r$ and a_i also gives me the same remainder r, so I can write a_i as some $q_i * n + r$.

Then if I take $a_j - a_i$ the effect of r cancels out and I get that its completely now a multiple of n. So, I showed you constructively here that irrespective of what is your n, I can always give you a number which is divisible by n and which has only 1's and 0's in its decimal expansion right.

(Refer Slide Time: 17:02)